

MOTIVIC MODEL CATEGORIES AND MOTIVIC DERIVED ALGEBRAIC GEOMETRY

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ABSTRACT. In this paper, we introduce the theory of motivic derived algebraic geometry which is obtained by combining Lurie’s derived algebraic geometry and Voevodsky’s \mathbb{A}^1 -homotopy theory. The theory of motivic derived algebraic geometry is established under the theory of motivic model categories. By using the theory of motivic model categories, we define motivic versions of ∞ -categories, ∞ -bicategories, ∞ -topoi and classifying ∞ -topoi. The main result of this paper is the existence of the spectrum functor Spec in the theory of motivic derived algebraic geometry. This result gives us the motivic versions of the formulations of spectral schemes and spectral Deligne–Mumford stacks introduced by Lurie.

1. INTRODUCTION

The theory of motivic derived algebraic geometry is an enhancement of derived algebraic geometry for the direction of \mathbb{A}^1 -homotopy theory. In the theory of motivic derived algebraic geometry, we define motivic versions of ∞ -categories, ∞ -topoi, classifying ∞ -topoi which are defined by Lurie [7] and [10]. We explain the theory of motivic derived algebraic geometry in Section 5, and prove an analogy of the existence of the spectrum functor (See [10, Theorem 2.1.1]).

In this paper, we introduce the theory of motivic model categories in order to establish the theory of motivic derived algebraic geometry. For any left proper combinatorial simplicial model category \mathbf{M} , the motivic model category $\mathrm{Mot}(\mathbf{M})$ is defined as a generalization of the definition of the model category of motivic spaces \mathbf{MS} introduced by Morel–Voevodsky [13]. In the case that \mathbf{M} is the model category of simplicial sets whose model structure is the Kan–Quillen model structure [14], the motivic model category coincides with the model category of motivic spaces. It is known that the model category \mathbf{MS} is left proper combinatorial simplicial symmetric monoidal model category by Jardine [12].

By Dugger’s representable theorem [4, Theorem 1.1] and the theory of ∞ -categories [7], we can prove that the 2-category $\mathrm{Model}_{\Delta}^{\mathrm{lpc}}$ of left proper combinatorial simplicial model categories

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has a model structure induced by the monoidal structure on the Model category Cat_Δ of simplicial categories. Here the model structure on Cat_Δ is the Dayer–Kan model structure introduced by Bergner [2]. Furthermore, $\text{Model}_\Delta^{\text{lpc}}$ has a monoidal structure which is commutative with the monoidal structure on Cat_Δ . In this view point, motivic model categories can be characterized as **MS**-module objects of $\text{Model}_\Delta^{\text{lpc}}$ (See Theorem 4.2).

By using the theory of motivic model categories, we define motivic versions of ∞ -categories, ∞ -bicategories and ∞ -topoi introduced by Lurie [7] and [8]. Roughly, these theories are obtained by replacing Kan complexes and motivic spaces.

In Section 5.3, we introduce a motivic version of the theory of classifying ∞ -topoi by using a motivic version of scaled straightening and unstraightening [8, p.114, Section 3.5] Theorem 5.4. The main theorem of this paper is Theorem 5.8 which is the existence of a motivic version of spectrum functor Spec . By using Theorem 5.8, we can formulate the motivic versions of spectral schemes and spectral Deligne–Mumford stacks in [10] and [11].

This paper is organized as follows: In Section 2.1, we explain that the 2-category of left proper simplicial model categories has the canonical model structure. In Section 3, following [8], we recall the definition of ∞ -bicategories and the scaled straightening and unstraightening theorem. These theory are applied to the construction of the theory of motivic derived algebraic geometry in Section 5. In Section 4, we introduce the theory of motivic model categories which is a generalization of the theory of motivic spaces introduced by Morel–Voevodsky [13]. In section 5, we introduce the theory of motivic derived algebraic geometry by combining the theory of motivic spaces (See [13] and [12]) with the theory of derived algebraic geometry [10] and [11]: We define motivic ∞ -spaces, motivic ∞ -categories, motivic ∞ -topoi and motivic classifying ∞ -topoi. In the final part of this paper, we prove the main theorem.

2. THE MODEL STRUCTURE ON THE 2-CATEGORY OF LEFT PROPER COMBINATORIAL SIMPLICIAL MODEL CATEGORIES.

2.1. The definition of combinatorial model categories. In this section, our main objects of model categories are left proper combinatorial simplicial model categories. We recall the definition of combinatorial model categories. Dugger [4] proved that for any combinatorial model category \mathbf{M} has a small presentation $\text{Rep}(\mathbf{M})$ which is a left proper combinatorial simplicial model category with a left Quillen equivalence $Re : \text{Rep}(\mathbf{M}) \rightarrow \mathbf{M}$. By using the theory of ∞ -categories [6] and [7], we obtain a model structure on the 2-category of left proper combinatorial model categories.

First, we recall the definition of model categories. In this paper, we assume that every model category is locally presentable.

Definition 2.1. A *model category* is a locally presentable category \mathbf{M} with a triple of subcategories $(\mathbf{W}_\mathbf{M}, \mathbf{C}_\mathbf{M}, \mathbf{F}_\mathbf{M})$, which satisfies the following axioms:

MC1 The category \mathbf{M} is stable under all small limits and colimits.

MC2 The class $\mathbf{W}_\mathbf{M}$ has the 2-out-of-3 property.

MC3 The three classes $\mathbf{W}_\mathbf{M}$, $\mathbf{C}_\mathbf{M}$ and $\mathbf{F}_\mathbf{M}$ of morphisms contain all isomorphisms and are closed under all retracts.

MC4 The class $\mathbf{F}_\mathbf{M}$ has the right lifting property with respect to all morphisms in the class $\mathbf{C}_\mathbf{M} \cap \mathbf{W}_\mathbf{M}$, and the class $\mathbf{F}_\mathbf{M} \cap \mathbf{W}_\mathbf{M}$ has the right lifting property with respect to all morphisms in the class $\mathbf{C}_\mathbf{M}$.

MC5 The couples $(\mathbf{C}_\mathbf{M} \cap \mathbf{W}_\mathbf{M}, \mathbf{F}_\mathbf{M})$ and $(\mathbf{C}_\mathbf{M}, \mathbf{F}_\mathbf{M} \cap \mathbf{W}_\mathbf{M})$ are functorial factorization systems.

A morphism in $\mathbf{W}_\mathbf{M}$, $\mathbf{C}_\mathbf{M}$ and $\mathbf{F}_\mathbf{M}$ is called a *weak equivalence*, a *cofibration* and a *fibration*, respectively. In addition, a morphism in the class $\mathbf{C}_\mathbf{M} \cap \mathbf{W}_\mathbf{M}$ and $\mathbf{F}_\mathbf{M} \cap \mathbf{W}_\mathbf{M}$ is called a *trivial cofibration* and a *trivial fibration*, respectively.

Let S be a collection of morphisms in a locally presentable category \mathbf{M} . Let ${}^\square S$ denote the set of morphisms in \mathbf{M} that it has the right lifting property with respect to all morphisms of S . Similarly, we let S^\square denote the set of morphisms in \mathbf{M} that it has the left lifting property with respect to all morphisms of S . We say that the set $({}^\square S)^\square$ is the *weakly saturated class* of morphisms generated by S .

Definition 2.2. Let \mathbf{M} be a model category. Let $\mathbf{W}_\mathbf{M}$ be the class of weak equivalences in \mathbf{M} and $\mathbf{C}_\mathbf{M}$ the class of cofibrations in \mathbf{M} . We say that \mathbf{M} is *combinatorial* if \mathbf{M} has two sets I and J that $\mathbf{C}_\mathbf{M}$ is the weakly saturated class of morphisms generated by I and $\mathbf{C}_\mathbf{M} \cap \mathbf{W}_\mathbf{M}$ is the weakly saturated class of morphisms generated by J . We say that a combinatorial model category \mathbf{M} is *tractable* if I can be chosen cofibrant domains.

If \mathbf{M} is a model category with the property that every object is cofibrant, then \mathbf{M} is tractable.

Definition 2.3. Let $F : \mathbf{M} \rightleftarrows \mathbf{N} : G$ be an adjunction between model categories. The adjunction $F : \mathbf{M} \rightleftarrows \mathbf{N} : G$ is called a Quillen adjunction if F and G preserve the factorization systems in the axiom MC5 of Definition 2.1. Then F and G are called a *left Quillen functor* and a *right Quillen functor*, respectively. Moreover, if the Quillen adjunction induces a categorical equivalence between the homotopy categories (See [14, Chapter 1].) of the model categories, then the Quillen adjunction is called a *Quillen equivalence*. Similarly, F and G are called a *left Quillen equivalence* and a *right Quillen equivalence*, respectively.

2.2. The model structure on the 2-category of left proper combinatorial model categories.

Dugger proved that any combinatorial model category has a small presentation:

Theorem 2.4 ([4] Theorem 1.1). *Let \mathbf{M} be a combinatorial model category. Then there exists a small category \mathcal{C} and a left Quillen functor $R : \text{Set}_\Delta^{\text{cop}} \rightarrow \mathbf{M}$ such that \mathbf{M} is a Quillen equivalent to a Bousfield localization of $\text{Set}_\Delta^{\text{cop}}$, where the model structure on $\text{Set}_\Delta^{\text{cop}}$ is the projective model structure.* □

In the proof of [4, Theorem 1.1], the small category \mathcal{C} is the full subcategory $\mathbf{M}_\lambda^{\text{cof}}$ of λ -compact cofibrant objects of \mathbf{M} for some regular cardinal λ (See [4, Section 5 and 6]). Hence every combinatorial model category is Quillen equivalent to some left proper simplicial combinatorial model category. Furthermore if the model category \mathbf{M} is symmetric monoidal, then the small presentation of \mathbf{M} is also symmetric monoidal and the Bousfield localization is a symmetric monoidal.

Let $\mathfrak{C} : \text{Set}_\Delta \rightleftarrows \text{Cat}_\Delta : N_\Delta$ denote the Quillen adjunction which is a Quillen equivalence between left proper combinatorial model categories (See [7, p.89, Theorem 2.2.5.1]). The model structure on the category Set_Δ of simplicial sets is the Joyal [6] model structure whose fibrant objects are ∞ -categories. The model structure on Cat_Δ is the Dayer–Kan model structure introduced by Bergner [2]. Let \mathbf{M} be a simplicial model category and \mathbf{M}° denote the full subcategory spanned by fibrant-cofibrant objects. Then \mathbf{M}° is a fibrant object of the model category Cat_Δ . Therefore the simplicial model category \mathbf{M} determines an ∞ -category $N_\Delta(\mathbf{M}^\circ)$. We call the ∞ -category $N_\Delta(\mathbf{M}^\circ)$ the *underlying ∞ -category* of \mathbf{M} . The following proposition gives a correspondence between locally presentable ∞ -categories and left proper combinatorial simplicial model categories. For any left proper combinatorial model category \mathbf{M} , we let $\text{Rep}(\mathbf{M})$ denote the small representation obtained by applying Theorem 2.4 to \mathbf{M} .

Lemma 2.5 (cf.[7] p.906, Remark A.3.7.7). *Let $F : \mathbf{M} \rightarrow \mathbf{N}$ be a left Quillen functor of left proper combinatorial simplicial model categories with the right adjoint $G : \mathbf{N} \rightarrow \mathbf{M}$. Then F is a left Quillen equivalence if and only if it induces an equivalence $N_\Delta(\mathbf{M}^\circ) \rightarrow N_\Delta(\mathbf{N}^\circ)$ of ∞ -categories.*

proof. Let λ be a regular cardinal such that the cofibrations and trivial cofibrations of the model categories \mathbf{M} and \mathbf{N} are generated by λ -filtered colimits. Let $\mathbf{M}_\lambda^{\text{cof}}$ denote the full subcategory of \mathbf{M} whose objects are λ -compact cofibrant objects. Write $\mathcal{C} = \mathbf{M}_\lambda^{\text{cof}}$. Then there exists a homotopically surjective left Quillen functor $R : \text{Set}_\Delta^{\text{cop}} \rightarrow \mathbf{M}$. Let $\text{Rep}(\mathbf{M})$ denote the Bousfield localization of \mathbf{M} which is Quillen equivalent to \mathbf{M} . Let $Re : \text{Rep}(\mathbf{M}) \rightarrow \mathbf{M}$ denote the induced left Quillen equivalence.

Suppose that F is a left Quillen equivalence. Then $F \circ Re$ and Re are left Quillen equivalences from $\text{Rep}(\mathbf{M})$ to \mathbf{N} and \mathbf{M} . Note that $\text{Rep}(\mathbf{M})$ satisfies the condition of [7, p.849, Proposition 3.1.10]. Hence we have weak equivalences of ∞ -categories $N_\Delta(\text{Rep}(\mathbf{M})^\circ) \simeq N_\Delta(\mathbf{M}^\circ)$ and $N_\Delta(\text{Rep}(\mathbf{M})^\circ) \simeq N_\Delta(\mathbf{N}^\circ)$ by [7, p.849, Proposition 3.1.10]. We assume that F induces an equivalence $N_\Delta(\mathbf{M}^\circ) \simeq N_\Delta(\mathbf{N}^\circ)$ of ∞ -categories. Then $F \circ Re$ induces a weak equivalence $N_\Delta(\text{Rep}(\mathbf{M})^\circ) \simeq N_\Delta(\mathbf{M}^\circ)$ of ∞ -categories. By the converse implication of [7, p.849, Proposition 3.1.10], $F \circ Re$ is a left Quillen equivalence. Hence the left Quillen functor $F : \mathbf{M} \rightarrow \mathbf{N}$ is homotopically surjective. By the constructions of the Bousfield localization $\text{Set}_\Delta^{\text{cop}} \rightarrow \text{Rep}(\mathbf{M})$ and the Quillen equivalence $Re : \text{Rep}(\mathbf{M}) \rightarrow \mathbf{M}$, we obtain that F is a left Quillen equivalence.

Definition 2.6. Given an adjunction between locally presentable categories

$$F : \mathbf{M} \rightleftarrows \mathbf{C} : G$$

where \mathbf{M} is a model category. We will define a model structure on \mathbf{C} by the following:

- (F) A morphism $f : X \rightarrow Y$ in \mathbf{C} is a fibration if $G(f) : G(X) \rightarrow G(Y)$ is a fibration in the model category \mathbf{M} .
- (W) A morphism $f : X \rightarrow Y$ in \mathbf{C} is a weak equivalence if $G(f) : G(X) \rightarrow G(Y)$ is a weak equivalence in the model category \mathbf{M} .
- (WF) A morphism $f : X \rightarrow Y$ in \mathbf{C} is a trivial fibration if $G(f) : G(X) \rightarrow G(Y)$ is a trivial fibration in the model category \mathbf{M} .
- (C) A morphism $f : X \rightarrow Y$ in \mathbf{C} is a cofibration if it has the right lifting property with respect to all trivial fibrations.

It is easily checked that \mathbf{C} is a model category. We say that the model structure of \mathbf{C} is the *projective model structure* induced by F .

Example 2.7. Let \mathcal{C} be a category and \mathbf{M} a model category. The diagonal functor $D : \mathbf{M} \rightarrow \mathbf{M}^{\mathcal{C}}$ has a right adjoint $\prod_{\mathcal{C}}$. The model structure of $\mathbf{M}^{\mathcal{C}}$ is said to be induced by \mathbf{M} .

The 2-category $\text{Model}_{\Delta}^{\text{lpc}}$ has a monoidal structure (it is not a monoidal model structure) which is commutative with the monoidal structure on $\text{Pr}_{\Delta}^{\text{L}}$. The model structure is defined as follows: For any two left proper simplicial combinatorial model categories \mathbf{M} and \mathbf{N} , the mapping model category $\text{Map}_{\text{Model}_{\Delta}^{\text{lpc}}}(\mathbf{M}, \mathbf{N})$ is defined by the projective model structure of $\mathbf{N}^{\mathbf{M}}$ induced by \mathbf{N} . By [7, p.829, Proposition A.2.8.2 and p.831 Remark A.2.8.4], it is known that $\mathbf{N}^{\mathbf{M}}$ is left proper combinatorial. Let $- \otimes -$ denote the monoidal structure on $\text{Model}_{\Delta}^{\text{lpc}}$. Then the model category Set_{Δ} is the unit object of $\text{Model}_{\Delta}^{\text{lpc}}$, where the model structure on Set_{Δ} is the Kan–Quillen model structure [14].

Dugger [3] proved that every functor $F : \mathcal{C} \rightarrow \mathbf{M}$ from a category to a model category factors through the model category $\text{Set}_{\Delta}^{\text{cop}}$ whose model structure is the projective model structure induced by Set_{Δ} . Furthermore the left Kan extension $F^+ : \text{Set}_{\Delta}^{\text{cop}} \rightarrow \mathbf{M}$ along F is a left Quillen functor. Moreover, if \mathbf{M} is simplicial then the left Quillen functor F^+ is simplicial. Write $U(\mathcal{C}) = \text{Set}_{\Delta}^{\text{cop}}$. Then $U(\mathcal{C})$ is called the universal model category of \mathcal{C} and $U : \text{Cat} \rightarrow \text{Model}_{\Delta}^{\text{lpc}}$ is called the *universal model category functor*.

The class of left Quillen equivalences of left proper combinatorial simplicial model categories coincides with the class of colimit preserving functors which induce weak equivalences of the underlying presentable ∞ -categories. Lurie’s theory of presentable ∞ -categories and Dugger’s representable theorem includes the following adjunction:

Theorem 2.8. Let Cat_{Δ} denotes the category of simplicial categories and $\text{Model}_{\Delta}^{\text{lpc}}$ the category of left proper combinatorial simplicial model categories. Then the universal model category

functor $U : \text{Cat}_\Delta \rightarrow \text{Model}_\Delta^{\text{lpc}}$ induces an adjunction

$$\overline{U} : \text{Pr}_\Delta^{\text{L}} \rightleftarrows \text{Model}_\Delta^{\text{lpc}} : G,$$

where $\text{Pr}_\Delta^{\text{L}}$ is the subcategory of Cat_Δ of locally presentable simplicial categories whose functors are colimit preserving functors and G sends simplicial model categories to the underlying simplicial categories. Then \overline{U} induces a model structure on $\text{Model}_\Delta^{\text{lpc}}$ which is the projective model structure induced by U . Furthermore \overline{U} is a left Quillen equivalence.

proof. By using Theorem 2.4 and Lemma 2.5, we have that the left Quillen functor $\overline{U} : \text{Pr}_\Delta^{\text{L}} \rightarrow \text{Model}_\Delta^{\text{lpc}}$ is a left Quillen equivalence.

3. THE DEFINITION OF ∞ -BICATEGORIES.

In this section, following [8], we explain the definition of ∞ -bicategories by using the theory of scaled simplicial sets and Set_Δ^+ -enriched categories. To establish motivic derived algebraic geometry, we use the ∞ -bicategorical straightening and unstraightening theorem.

3.1. Scaled simplicial sets and Set_Δ^+ -enriched categories.

Definition 3.1 ([8] p.66, Definition 3.1.1). A *scaled simplicial set* $\overline{X} = (X, T)$ is a pair, where X is a simplicial set and T is a set 2-simplex of X which contains all degenerate 2-simplex of X . We call the simplicial set X the underlying simplicial set and the elements of T *thin*. Let (X, T) and (X', T') be scaled simplicial sets. A morphism from (X, T) to (X', T') is a map $f : X \rightarrow X'$ of simplicial sets which T carries into T' . We let $\text{Set}_\Delta^{\text{sc}}$ denote the category of scaled simplicial sets.

Let Set_Δ^+ denote the simplicial model category of marked simplicial sets. The model structure on Set_Δ^+ is Cartesian model structure. Furthermore Set_Δ^+ is a monoidal model category. Let Cat_Δ^+ denote the category of Set_Δ^+ -enriched categories. Then the category Cat_Δ^+ has a model structure which is induced by the Cartesian model structure on Set_Δ^+ . We explain the induced model structure on Cat_Δ^+ : Let X be a marked simplicial set. Then we define a new Set_Δ^+ -enriched category $[1]_X$ as follows:

- The category $[1]$ has only two objects 0 and 1.
- The marked simplicial set of morphisms $\text{Hom}_{[1]_X}(x, y)$ is defined by the formula:

$$\text{Hom}_{[1]_X}(x, y) = \begin{cases} \Delta_b^0 & (x = y), \\ X & (x = 0, y = 1), \\ \emptyset & (x = 1, y = 0). \end{cases}$$

Definition 3.2 (c.f. [7] p.856, Definition A.3.2.1). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor of Set_Δ^+ -enriched categories. We say that the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a weak equivalence if the following conditions are satisfied:

(1) For any $X, Y \in \mathcal{C}$, the induced map

$$\text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{\mathcal{D}}(F(X), F(Y))$$

is a Cartesian equivalence of marked simplicial sets.

(2) The functor F induces an essential surjective functor between their homotopy categories.

Definition 3.3 ([7] p.857, Definition A.3.2.4). The category Cat_{Δ}^+ is a model category defined by the following:

- (W) Weak equivalences are functors satisfying the conditions in Definition 3.2.
- (C) Cofibrations are morphisms in the smallest weakly saturated class [7, p.783, Definition A.1.2.2] of morphisms containing the following collection of morphisms:
 - The inclusion $\emptyset \rightarrow [0]$, where \emptyset is the empty category and $[0]$ is the category which consists of the single object Δ_b^0 .
 - The induced map $[1]_X \rightarrow [1]_Y$, where $X \rightarrow Y$ is a morphism which belongs to the weakly saturated class of cofibrations in Set_{Δ}^+ .
- (F) Fibrations are morphisms which have right lifting property with respect to all morphisms satisfying both conditions (W) and (C).

Following [8, pp.69–70, Definition 3.1.10], we define a functor $N^{\text{sc}} : \text{Cat}_{\Delta}^+ \rightarrow \text{Set}_{\Delta}^{\text{sc}}$ as follows:

- For $\mathcal{C} \in \text{Cat}_{\Delta}^+$, the underlying simplicial set is the simplicial nerve $N_{\Delta}(\mathcal{C})$ of \mathcal{C} .
- Given a 2-simplex σ of $N_{\Delta}(\mathcal{C})$ corresponding to a (not necessary commutative) diagram

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z \end{array}$$

in \mathcal{C} and an edge $\alpha : h \mapsto g \circ f$ of the marked simplicial set $\text{Map}_{\mathcal{C}}(X, Z)$, we say that σ is thin if α is a marked edge.

We call the functor $N^{\text{sc}} : \text{Cat}_{\Delta}^+ \rightarrow \text{Set}_{\Delta}^{\text{sc}}$ the *scaled nerve functor*. A right adjoint functor $\mathfrak{C}^{\text{sc}} : \text{Set}_{\Delta}^{\text{sc}} \rightarrow \text{Cat}_{\Delta}^+$ of the scaled nerve functor is defined as follows:

- For any scaled simplicial set $\overline{S} = (S, T)$, the underlying simplicial category of $\mathfrak{C}^{\text{sc}}[\overline{S}]$ is the simplicial category $\mathfrak{C}[S]$.
- Given $x, y \in S$, an edge α of the simplicial set $\text{Map}_{\mathfrak{C}^{\text{sc}}[\overline{S}]}(x, y)$ is a marked edge if there exist a sequence $x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = y$ of vertexes and a sequence of thin 2-simplices

$$\begin{array}{ccc} & y_i & \\ f_i \nearrow & & \searrow g_i \\ x_{i-1} & \xrightarrow{h_i} & x_i \end{array}$$

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of S such that $\alpha = \alpha_n \circ \cdots \circ \alpha_1$ where $\alpha_i : h_i \mapsto g_i \circ f_i$ is an edge of the simplicial set $\text{Map}_{\mathbb{C}[S]}(x_{i-1}, x_i)$ for each $1 \leq i \leq n$.

Then the pair $(\mathbb{C}^{\text{sc}}, N^{\text{sc}})$ of functors determines an adjunction

$$\mathbb{C}^{\text{sc}} : \text{Set}_{\Delta}^{\text{sc}} \rightleftarrows \text{Cat}_{\Delta}^+ : N^{\text{sc}}.$$

Moreover, it is known that the adjunction is a Quillen adjunction by [8, p.71, Proposition 3.1.13].

The model category $\text{Set}_{\Delta}^{\text{sc}}$ of scaled simplicial sets is left proper combinatorial. By Dugger's representable theorem [4] there exists a left proper combinatorial simplicial model category $\text{Rep}(\text{Set}_{\Delta}^{\text{sc}})$ such that the realization functor $Re : \text{Rep}(\text{Set}_{\Delta}^{\text{sc}}) \rightarrow \text{Set}_{\Delta}^{\text{sc}}$ is a left Quillen equivalence.

3.2. Scaled straightening and unstraightening. We explain the definition of the scaled straightening functor and unstraightening functor by the following [8, Section 3]. Let $\overline{S} = (S, T)$ be a scaled simplicial set and \mathcal{C} a Set_{Δ}^+ -enriched category. Given a functor $\phi : \mathbb{C}^{\text{sc}}[\overline{S}] \rightarrow \mathcal{C}$, we define the scaled straightening functor $\text{St}_{\phi}^{\text{sc}} : \text{Set}_{\Delta/\overline{S}}^+ \rightarrow (\text{Set}_{\Delta}^+)^{\mathcal{C}}$.

Definition 3.4 ([8] p.114, Definition 3.5.1). Let $\overline{X} = (X, M)$ be a marked simplicial set. Let $T \subset X \times \Delta^1$ be the collection of all 2-simplices σ with the following properties:

- For the projection $X \times \Delta^1 \rightarrow X$, the image of σ is a degenerate 2-simplex of X .
- For any 2-simplex $\pi : \Delta^2 \xrightarrow{\sigma} X \times \Delta^1 \rightarrow \Delta^1$ satisfying $\pi^{-1}(\{0\}) = \Delta^{0,1}$, the image of π of the restriction $\sigma|_{\Delta^{0,1}}$ determines a marked edge of X .

We define a scaled simplicial set $C(\overline{X})$ by the following formula:

$$C(\overline{X}) = (X \times \Delta^1) \coprod_{(X \times \{0\})_b} \{v\}_b.$$

We call $C(\overline{X})$ the *scaled cone* of \overline{X} . More generally, for any scaled simplicial set \overline{S} and $\overline{X} \in (\text{Set}_{\Delta}^{\text{sc}})_{/\overline{S}}$, we set $C_{\overline{S}}(\overline{X}) = C(\overline{X}) \coprod_{(X \times \{1\})_b} \overline{S}$. We say that $C_{\overline{S}}(\overline{X})$ is the *scaled cone* of \overline{X} over \overline{S} .

Definition 3.5 ([8] p.115, Definition 3.5.4). Let \overline{S} be a scaled simplicial set, \mathcal{C} a Set_{Δ}^+ -enriched category and $\phi : \mathbb{C}^{\text{sc}}[\overline{S}] \rightarrow \mathcal{C}$ a functor of Set_{Δ}^+ -enriched categories. We define the *scaled straightening functor* associated to ϕ $\text{St}_{\phi}^{\text{sc}} : \text{Set}_{\Delta/\overline{S}}^+ \rightarrow (\text{Set}_{\Delta}^+)^{\mathcal{C}}$ of \overline{X} by the following:

$$(\text{St}_{\phi}^{\text{sc}}(\overline{X}))(C) = \text{Map}_{C_{\overline{S}}[\overline{X}] \coprod_{\mathbb{C}^{\text{sc}}[\overline{S}]} \mathcal{C}}(v, C),$$

for any $C \in \mathcal{C}$.

Remark 3.6. The straightening functor $\text{St}_{\phi}^{\text{sc}} : \text{Set}_{\Delta/\overline{S}}^+ \rightarrow (\text{Set}_{\Delta}^+)^{\mathcal{C}}$ is defined as a Set_{Δ}^+ -enriched categorical colimit. Let $f : X \rightarrow S$ be a marked simplicial set over S . Then the straightening $\text{St}_{\phi}^{\text{sc}}(\overline{X})$ is equivalent to the colimit of the diagram:

$$j^{\text{op}} \circ \phi \circ \mathbb{C}^{\text{sc}}[F] : \mathbb{C}^{\text{sc}}[(X \times \Delta^1, T) \coprod_{X \times \{1\}} \overline{S}] \rightarrow \mathbb{C}^{\text{sc}}[\overline{S}] \rightarrow \mathcal{C} \rightarrow (\text{Set}_{\Delta}^+)^{\mathcal{C}}$$

where F is an induced map by f and $j : \mathcal{C} \rightarrow (\text{Set}_\Delta^+)^{\text{cop}}$ denotes the enriched Yoneda embedding.

Since the scaled straightening functor $\text{St}_\phi^{\text{sc}}$ preserves all small colimits, it has a right adjoint functor by the adjoint functor theorem for categories, and $\text{Un}_\phi^{\text{sc}}$ denotes the right adjoint functor. We call the right adjoint $\text{Un}_\phi^{\text{sc}}$ the *scaled unstraightening functor* associated to ϕ . It is known that the adjunction $(\text{St}_\phi^{\text{sc}}, \text{Un}_\phi^{\text{sc}})$ is a Quillen adjunction. Here the model structure on $\text{Set}_{\Delta/\bar{S}}^+$ is the locally coCartesian model structure [8, p.74, Example 3.2.9] and the model structure on $(\text{Set}_\Delta^+)^{\mathcal{C}}$ is the projective model structure [7, pp.823–824, Definition A.2.8.1 and Proposition A.2.8.2].

We recall the scaled straightening and unstraightening theorem:

Theorem 3.7 ([8] p.128, Theorem 3.8.1). *Let \bar{S} be a scaled simplicial set, \mathcal{C} a Set_Δ^+ -enriched category and $\phi : \mathcal{C}^{\text{sc}}[\bar{S}] \rightarrow \mathcal{C}$ a weak equivalence of Set_Δ^+ -enriched categories. Then the Quillen adjunction*

$$\text{St}_\phi^{\text{sc}} : \text{Set}_{\Delta/\bar{S}}^+ \rightleftarrows (\text{Set}_\Delta^+)^{\mathcal{C}} : \text{Un}_\phi^{\text{sc}}$$

is a Quillen equivalence. □

3.3. Definition of ∞ -bicategories. We explain the definition of a model structure on $\text{Set}_\Delta^{\text{sc}}$ by using the model structure on Cat_Δ^+ .

Definition 3.8 ([8] p.115, Definition 3.5.6). Let $f : \bar{X} \rightarrow \bar{Y}$ be a morphism of scaled simplicial sets. We say that the morphism $f : \bar{X} \rightarrow \bar{Y}$ is a *bicategorical equivalence* if the induced functor $\mathcal{C}^{\text{sc}}[f] : \mathcal{C}^{\text{sc}}[\bar{X}] \rightarrow \mathcal{C}^{\text{sc}}[\bar{Y}]$ is a weak equivalence of Set_Δ^+ -enriched categories.

Theorem 3.9 ([8] p.143, Theorem 4.2.7). *Let $\text{Set}_\Delta^{\text{sc}}$ denote the category of scaled simplicial sets. Then $\text{Set}_\Delta^{\text{sc}}$ has a left proper combinatorial model structure defined by the following:*

- (W) *The weak equivalences in $\text{Set}_\Delta^{\text{sc}}$ are bicategorical equivalences.*
 - (C) *The cofibrations in $\text{Set}_\Delta^{\text{sc}}$ are monomorphisms.*
 - (F) *The fibrations in $\text{Set}_\Delta^{\text{sc}}$ are morphisms which have the right lifting property with respect to all morphisms satisfying (W) and (C).*
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We say that the model structure on $\text{Set}_\Delta^{\text{sc}}$ in Theorem 3.9 is the *scaled model structure*.

Definition 3.10 ([8] p.145, Definition 4.2.8). An ∞ -bicategory is a fibrant object of $\text{Set}_\Delta^{\text{sc}}$ with respect to the scaled model structure.

Remark 3.11. The model category $\text{Set}_\Delta^{\text{sc}}$ is Cartesian closed, but I do not know that it has a simplicial model structure. Since $\text{Set}_\Delta^{\text{sc}}$ is left proper combinatorial and Cartesian closed, we have a canonical left Quillen equivalence $Re : \text{Rep}(\text{Set}_\Delta^{\text{sc}}) \rightarrow \text{Set}_\Delta^{\text{sc}}$ such that $\text{Rep}(\text{Set}_\Delta^{\text{sc}})$ is left proper combinatorial simplicial monoidal model category. On the other hand, Lurie [8] proved

that there is a right Quillen equivalence $F : (\text{Set}_\Delta^+)_{/N(\Delta^{\text{op}})} \rightarrow \text{Set}_\Delta^{\text{sc}}$ where the model structure of $(\text{Set}_\Delta^+)_{/N(\Delta^{\text{op}})}$ is the complete Segal model structure which is simplicial and monoidal.

4. MOTIVIC MODEL CATEGORIES.

In this section, we introduce the theory of motivic model categories for left proper combinatorial simplicial model categories.

4.1. Definition of motivic model structure of a left proper combinatorial simplicial model category. Let S be a regular Noetherian separated scheme of finite dimension. Let \mathbf{M} be a left proper combinatorial simplicial model category. Let $\mathbf{M}^{(\mathbf{Sm}_S)_{\text{Nis}}^{\text{op}}}$ be the functor category from the Nisnevich site $(\mathbf{Sm}_S)_{\text{Nis}}$. We write $\text{Mot}(\mathbf{M}) = \mathbf{M}^{(\mathbf{Sm}_S)_{\text{Nis}}^{\text{op}}}$. We define a new model structure on $\text{Mot}(\mathbf{M})$ by the following: Let $f : X \rightarrow Y$ be a map of objects of \mathbf{M} . We say that f is a *stalk-wise weak equivalence* if the induced morphism on stalks $f_x : X_x \rightarrow Y_x$ is a weak equivalence of the model category \mathbf{M} for each point x of S . A *cofibration* is a pointwise cofibrations of \mathbf{M} . A *trivial stalk-wise cofibration* is a map of objects of $\text{Mot}(\mathbf{M})$ in which is a stalk-wise equivalence and a cofibration. A *global fibration* is a map of simplicial sheaves which has a right lifting property with respect to all trivial stalk-wise cofibrations. If $X \rightarrow *$ is a global fibration, then we say that X is *globally fibrant*. Let X be an object of $\text{Mot}(\mathbf{M})$. Then X is \mathbb{A}^1 -*local* if the induced map $X(U \times \mathbb{A}^1) \rightarrow X(U)$ is a weak equivalence in \mathbf{M} for any smooth schemes U over S . A motivic fibrant object is a globally fibrant and \mathbb{A}^1 -local. A map $f : X \rightarrow Y$ in $\text{Mot}(\mathbf{M})$ is a *motivic \mathbf{M} -equivalence* if the induced map

$$f^* : \text{Hom}_{\text{Mot}(\mathbf{M})}(Y, Z) \rightarrow \text{Hom}_{\text{Mot}(\mathbf{M})}(X, Z)$$

is a weak homotopy equivalence of simplicial sets for each motivic fibrant object Z of $\text{Mot}(\mathbf{M})$. We call the modal category $\text{Mot}(\mathbf{M})$ is the *motivic model category* of \mathbf{M} .

Theorem 4.1. *Let \mathbf{M} be a left proper combinatorial simplicial model category. There is a left proper combinatorial simplicial model structure on $\text{Mot}(\mathbf{M})$ by the following:*

- (C) *Cofibrations are pointwise cofibrations.*
- (W) *Weak equivalences are motivic \mathbf{M} -weak equivalences.*
- (F) *Fibrations are morphisms which has a left lifting property with respect to all morphisms which are both cofibrations and motivic \mathbf{M} -weak equivalences.*

Furthermore, if \mathbf{M} is symmetric monoidal, then the model category $\text{Mot}(\mathbf{M})$ is also symmetric monoidal.

proof. By [1, p.56, Corollary 4.55], the model category $\mathbf{M}^{(\mathbf{Sm}_S)_{\text{Nis}}^{\text{op}}}$ is a left proper combinatorial simplicial monoidal model category, whose model structure is the projective model structure. The category $\text{Mot}(\mathbf{M})$ is a Bousfield localization of the monoidal model category. Hence $\text{Mot}(\mathbf{M})$ is also a proper combinatorial simplicial model category. By [1, p.54 Proposition 4.47],

the Bousfield localization $\text{Loc}_{\mathbb{A}^1} : \mathbf{M}^{(\text{Sm})_{\text{Nis}}^{\text{op}}} \rightarrow \text{Mot}(\mathbf{M})$ is a symmetric monoidal localization. Hence $\text{Mot}(\mathbf{M})$ is also a symmetric monoidal model category.

If $\mathbf{M} = \text{Set}_{\Delta}$ with the Kan–Quillen model structure, we call the fibrant object of $\text{Mot}(\mathbf{M})$ a *motivic spaces* and the ∞ -category $N_{\Delta}(\text{Mot}(\text{Set}_{\Delta})^{\circ})$ the ∞ -category of *motivic spaces*. If $\mathbf{M} = \text{Set}_{\Delta}^{+}$ with the Cartesian model structure, then we say that $N_{\Delta}(\text{Mot}(\text{Set}_{\Delta}^{+})^{\circ})$ is the ∞ -category of *motivic ∞ -category*. Moreover, we say that the ∞ -bicategory $N^{\text{sc}}(\text{Mot}(\text{Set}_{\Delta}^{+})^{\circ})$ is the ∞ -bicategory of motivic ∞ -categories. We write $\mathbf{MS}_{\infty} = N_{\Delta}(\text{Mot}(\text{Set}_{\Delta})^{\circ})$, $\mathbf{MCat}_{\infty} = N_{\Delta}(\text{Mot}(\text{Set}_{\Delta}^{+})^{\circ})$ and $\mathbf{MCat}_{\infty} = N^{\text{sc}}(\text{Mot}(\text{Set}_{\Delta}^{+})^{\circ})$. The symmetric monoidal structures on Set_{Δ}^{+} induces the symmetric monoidal structure on $\mathbf{MCat}_{\infty} = N_{\Delta}(\text{Mot}(\text{Set}_{\Delta}^{+})^{\circ})$. Hence \mathbf{MCat}_{∞} is a symmetric monoidal ∞ -category.

4.2. The universal property of $\text{Mot}(\mathbf{M})$. Let \mathbf{M} be a left proper combinatorial simplicial model category. The motivic model category \mathbf{M} has the following universal property.

Theorem 4.2. *Let S be a regular Noetherian separated scheme of finite dimension. Let $\text{Model}_{\Delta}^{\text{lpc}}$ denote the 2-category of left proper combinatorial simplicial model categories, whose functors are left Quillen functors. Let \mathbf{M} be a left proper combinatorial simplicial model category. Write $\mathbf{MS} = \text{Mot}(\text{Set}_{\Delta})$. Then the left Quillen functor $D \otimes \text{Mot}(\mathbf{1}) : \mathbf{M} \otimes \mathbf{MS} \rightarrow \text{Mot}(\mathbf{M})$ is a left Quillen equivalence, where $D : \mathbf{M} \rightarrow \text{Mot}(\mathbf{M})$ denotes the diagonal functor.*

proof. By Theorem 2.4, it is sufficient to prove the theorem in the case $\mathbf{M} = \text{Set}_{\Delta}^{\mathcal{C}}$ with a simplicial category \mathcal{C} . The 2-category $\text{Model}_{\Delta}^{\text{lpc}}$ has a Cartesian closed monoidal structure with the unit object Set_{Δ} . The unit map $\mathbf{1} : \text{Set}_{\Delta} \rightarrow \text{Set}_{\Delta}^{\mathcal{C}}$ induces the left Quillen functor $\text{Mot}(\mathbf{1}) : \mathbf{MS} \rightarrow \text{Mot}(\text{Set}_{\Delta}^{\mathcal{C}})$. Here the unit map $\mathbf{1} : \text{Set}_{\Delta} \rightarrow \text{Set}_{\Delta}^{\mathcal{C}}$ is the diagonal functor. Furthermore there is a canonical equivalence $\text{Mot}(\text{Set}_{\Delta}^{\mathcal{C}}) \simeq \text{Mot}(\text{Set}_{\Delta})^{\mathcal{C}} = \mathbf{MS}^{\mathcal{C}}$. By Theorem 2.8, we have a chain of equivalences $\mathbf{MS}_{\infty} \otimes N_{\Delta}(\text{Set}_{\Delta}^{\mathcal{C}}) \simeq \text{Fun}^{\text{L}}(\mathbf{MS}_{\infty}^{\text{op}}, \mathcal{S})^{N_{\Delta}(\mathcal{C})} \simeq \mathbf{MS}_{\infty}^{N_{\Delta}(\mathcal{C})} \simeq N_{\Delta}(\text{Mot}(\text{Set}_{\Delta})^{\mathcal{C}})$. Hence the canonical functor $\mathbf{MS} \otimes \mathbf{M} \rightarrow \mathbf{M}$ is a left Quillen equivalence.

Corollary 4.3. *The left proper combinatorial simplicial model category \mathbf{M} is a motivic model category if and only if the underlying ∞ -category is a \mathbf{MS}_{∞} -module object of Pr^{L} . Here Pr^{L} denotes the symmetric monoidal ∞ -category of presentable ∞ -categories whose functors are colimit preserving functors.* \square

5. MOTIVIC DERIVED ALGEBRAIC GEOMETRY.

5.1. The motivic ∞ -category of motivic spaces and the motivic ∞ -category of motivic ∞ -categories. Let \mathbf{M} be a left proper combinatorial simplicial monoidal model category. If \mathbf{M} has a Cartesian closed symmetric monoidal model structure, then the model category $\text{Mot}(\mathbf{M})$ is an \mathbf{M} -enriched model category. Therefore $\text{Mot}(\mathbf{M})$ is tensored over \mathbf{M} , and we have a left Quillen

bifunctor

$$- \otimes - : \mathbf{M} \otimes \text{Mot}(\mathbf{M}) \rightarrow \text{Mot}(\mathbf{M}).$$

Let $\mathbf{1}$ be the unit element of the monoidal model category $\text{Mot}(\mathbf{M})$. Then the left Quillen bifunctor $- \otimes -$ induces a Quillen adjunction

$$- \otimes \mathbf{1} : \mathbf{M} \rightleftarrows \text{Mot}(\mathbf{M}) : \text{Hom}_{\text{Mot}(\mathbf{M})}(\mathbf{1}, -).$$

By using the above Quillen adjunction, we define the motivic ∞ -category of motivic spaces, the motivic ∞ -category of motivic ∞ -categories and the motivic ∞ -category of motivic ∞ -topoi. Let $\mathbf{1}$ be the unit element of the symmetric monoidal ∞ -category \mathbf{MCat}_∞ . Let \mathcal{S} denote the ∞ -category of spaces, Cat_∞ the ∞ -category of ∞ -categories and ${}^L\mathcal{T}\text{op}$ the ∞ -category of ∞ -topoi. By using Theorem 4.2, we have $\mathbf{MS}_\infty = \mathcal{S} \otimes \mathbf{1}$ and $\mathbf{MCat}_\infty = \text{Cat}_\infty \otimes \mathbf{1}$. We say that \mathbf{MS} is the motivic ∞ -category of motivic spaces and \mathbf{MCat}_∞ is the motivic ∞ -category of motivic ∞ -categories. We set ${}^L\mathbf{MTop} = {}^L\mathcal{T}\text{op} \otimes \mathbf{1}$, and we say that ${}^L\mathbf{MTop}$ is the motivic ∞ -category of *motivic ∞ -topoi* and an object of ${}^L\mathbf{MTop}$ is a motivic ∞ -topoi.

5.2. Definition of motivic ∞ -bicategories. The model category of scaled simplicial sets $\text{Set}_\Delta^{\text{sc}}$ is left proper combinatorial. However this is not simplicial or monoidal. In order to formulate the motivic model category of $\text{Set}_\Delta^{\text{sc}}$, we use the model category $(\text{Set}_\Delta^+)^{N(\Delta)^{\text{op}}}$ which is left proper combinatorial simplicial symmetric monoidal category. The model structure on $(\text{Set}_\Delta^+)^{N(\Delta)^{\text{op}}}$ is the Bousfield localization of the coCartesian model structure induced by the complete Segal model structure on $(\text{Set}_\Delta^+)^{\Delta^{\text{op}}}$ [8, p.34, Proposition 1.5.7]. There is a left Quillen functor $\text{sd}^+ : \text{Set}_\Delta^{\text{sc}} \rightarrow (\text{Set}_\Delta^+)^{N(\Delta)^{\text{op}}}$ which is called a *subdivision* functor [8, p.145, Definition 4.3.1]. By [8, p.150, Theorem 4.3.1.13], the subdivision functor $\text{sd}^+ : \text{Set}_\Delta^{\text{sc}} \rightarrow (\text{Set}_\Delta^+)^{N(\Delta)^{\text{op}}}$ is a left Quillen equivalence.

Proposition 5.1. *Let $\text{Mot}(\text{Set}_\Delta^{\text{sc}})$ denote the functor category $(\text{Set}_\Delta^{\text{sc}})^{(\mathbf{Sm}/S)^{\text{op}}}$ and $\text{Mot}(\text{sd}^+) : \text{Mot}(\text{Set}_\Delta^{\text{sc}}) \rightleftarrows \text{Mot}((\text{Set}_\Delta^+)^{N(\Delta)^{\text{op}}}) : \text{Mot}(F)$ the adjunction induced by the Quillen equivalence $\text{sd}^+ : \text{Set}_\Delta^{\text{sc}} \rightleftarrows (\text{Set}_\Delta^+)^{N(\Delta)^{\text{op}}} : F$. We define a model structure on $\text{Mot}(\text{Set}_\Delta^{\text{sc}})$ as follows:*

- (C) *A morphism $f : \overline{X} \rightarrow \overline{Y}$ is a cofibration if and only if it is a pointwise cofibration.*
- (W) *A morphism $f : \overline{X} \rightarrow \overline{Y}$ is a weak equivalence if and only if it $\text{Mot}(\text{sd}^+)$ is a motivic $(\text{Set}_\Delta^+)^{N(\Delta)^{\text{op}}}$ -equivalence.*
- (F) *A morphism $f : \overline{X} \rightarrow \overline{Y}$ is a fibration if and only if it has the right lifting property with respect to all morphisms which satisfies the condition (C) and (W).*

Then the Quillen adjunction $\text{Mot}(\text{sd}^+) : \text{Mot}(\text{Set}_\Delta^{\text{sc}}) \rightleftarrows \text{Mot}((\text{Set}_\Delta^+)^{N(\Delta)^{\text{op}}}) : \text{Mot}(F)$ is a Quillen equivalence.

proof. By the definition of the model structure on $\text{Mot}(\text{Set}_\Delta^{\text{sc}})$, the induced functor $\text{Mot}(\text{sd}^+)$ is a left Quillen functor. Note that the functor $(-)^{(\mathbf{Sm}/S)^{\text{op}}}$ preserves left Quillen equivalence between

the left proper combinatorial model categories where the functor $(-)^{(\mathbf{Sm}/S)^{\text{op}}}$ induces the projective model structure. The model structure on $\text{Mot}(\text{Set}_{\Delta}^{\text{sc}})$ is just the Bousfield localization of $(\text{Set}_{\Delta}^{\text{sc}})^{(\mathbf{Sm}/S)^{\text{op}}}$ that the Quillen adjunction $\text{Mot}(\text{sd}^+) : \text{Mot}(\text{Set}_{\Delta}^{\text{sc}}) \rightleftarrows \text{Mot}((\text{Set}_{\Delta}^+)_{/N(\Delta)^{\text{op}}}) : \text{Mot}(F)$ is a Quillen equivalence.

Definition 5.2. We say that a fibrant object of the model category $\text{Mot}(\text{Set}_{\Delta}^{\text{sc}})$ is a *motivic ∞ -bicategory*

5.3. The motivic scaled straightening and unstraightening. Let \mathbf{M} be a left proper combinatorial simplicial monoidal model category and $Re : \text{Rep}(\mathbf{M}) \rightleftarrows \mathbf{M} : G$ a small presentation. Let X be an object of \mathbf{M} . Then the small presentation induces an adjunction

$$Re_{/X} : \text{Rep}(\mathbf{M})_{/G(X)} \rightleftarrows \mathbf{M}_{/X} : G_{/X},$$

where the left is a left proper combinatorial model category on which model structure is the projective model structure induced by the covariant model structure [7, p.68, Definition 2.1.4.5] under the equivalence $(\text{Set}_{\Delta}^{\text{M}^{\text{op}}})_{/G(X)} = \int_{c \in \mathbf{M}_{\lambda}^{\text{op}}} (\text{Set}_{\Delta})_{/G(X)(c)}$ of model categories. We have a model structure on $\mathbf{M}_{/X}$ which is the projective model structure induced by $\text{Rep}(\mathbf{M})_{/G(X)}$. We say that the model structure on $\mathbf{M}_{/X}$ is the *covariant model structure*.

Lemma 5.3. Let \mathbf{M} be a left proper combinatorial simplicial monoidal model category. Let X be an object of \mathbf{M} and \mathcal{C} a simplicial category. Then $- \otimes \mathbf{1} : \mathbf{M} \rightarrow \text{Mot}(\mathbf{M})$ induces left Quillen equivalences

$$\text{Mot}(\mathbf{M}_{/X}) \rightarrow \text{Mot}(\mathbf{M})_{/X \otimes \mathbf{1}}, \text{Mot}(\mathbf{M}^{\mathcal{C}}) \rightarrow \text{Mot}(\mathbf{M})^{\mathcal{C}},$$

where the model structures on $\mathbf{M}_{/X}$ and $\text{Mot}(\mathbf{M})_{/X \otimes \mathbf{1}}$ are covariant model structures and the model structures on $\mathbf{M}^{\mathcal{C}}$ and $\text{Mot}(\mathbf{M})^{\mathcal{C}}$ are projective model structures.

proof. By Lemma 2.5, it is sufficient to prove that we have weak equivalences $N_{\Delta}(\text{Mot}(\mathbf{M}_{/X})^{\circ}) \rightarrow N_{\Delta}(\text{Mot}((\mathbf{M})_{/X \otimes \mathbf{1}})^{\circ})$ and $N_{\Delta}((\text{Mot}(\mathbf{M}^{\mathcal{C}}))^{\circ}) \rightarrow N_{\Delta}((\text{Mot}(\mathbf{M})^{\mathcal{C}})^{\circ})$ of ∞ -categories. It is clear that the second induced functor is a weak equivalence. Since these ∞ -categories $N_{\Delta}(\text{Mot}(\mathbf{M}_{/X})^{\circ})$ and $N_{\Delta}(\text{Mot}((\mathbf{M})_{/X \otimes \mathbf{1}})^{\circ})$ are equivalent to the ∞ -category $\text{Fun}(X^{\text{op}}, N_{\Delta}(\text{Mot}(\mathbf{M})^{\circ}))$, the first induced functor is a weak equivalence.

Let \overline{X} be a fibrant scaled simplicial set. Since the subdivision functor $\text{sd}^+ : \text{Set}_{\Delta}^{\text{sc}} \rightarrow (\text{Set}_{\Delta}^+)_{/N(\Delta)^{\text{op}}}$ is a left Quillen equivalence, there exists a fibrant object X^+ of $(\text{Set}_{\Delta}^+)_{/N(\Delta)^{\text{op}}}$ such that \overline{X} is weakly equivalent to $F(X^+)$. Let $- \otimes \mathbf{1} : (\text{Set}_{\Delta}^+)_{/N(\Delta)^{\text{op}}} \rightarrow \text{Mot}((\text{Set}_{\Delta}^+)_{/N(\Delta)^{\text{op}}})$ denote the functor defined Section 5.1, and write $\overline{X} \otimes \mathbf{1} = F(X^+ \otimes \mathbf{1})$. Then we have a motivic version of the straightening and unstraightening:

Theorem 5.4. Let \mathcal{C} be a Set_{Δ}^+ -enriched category. Then the induced Quillen adjunction the motivic scaled straightening and unstraightening:

$$\text{Mot}(\text{St}_{\mathcal{C}}^{\text{sc}}) : \text{Mot}(\text{Set}_{\Delta}^+)_{/N^{\text{sc}}(\mathcal{C}) \otimes \mathbf{1}} \rightleftarrows \text{Mot}(\text{Set}_{\Delta}^+)^{\mathcal{C}^{\text{op}}} : \text{Mot}(\text{Un}_{\mathcal{C}}^{\text{sc}}).$$

is a Quillen equivalence.

proof. This is a direct result from Lemma 5.3 and the scaled straightening and unstraightening: Theorem 3.7.

5.4. Motivic ∞ -topoi and motivic classifying ∞ -topoi. Let \mathcal{X} be a motivic ∞ -category. We say that \mathcal{X} is a motivic ∞ -topos if there exists an ∞ -topos \mathcal{X}_0 such that $\mathcal{X} \simeq \mathcal{X}_0 \otimes \mathbf{MS}_\infty$. Equivalently, a motivic ∞ -topos is an \mathbf{MS}_∞ -module object of the symmetric monoidal ∞ -category of ∞ -topoi.

Proposition 5.5. *Let \mathcal{X} be a motivic ∞ -topos. Then \mathcal{X} is also an ∞ -topos.*

proof. Let \mathcal{C} be a small motivic ∞ -category. Then \mathcal{C} is also an ∞ -category. Then there exists a simplicial category \mathcal{D} and a weak equivalence $\mathfrak{C}[\mathcal{C} \times \mathbf{MS}^{\omega, \text{op}}] \rightarrow \mathcal{D}$ of simplicial categories such that \mathcal{X} is an accessible left exact localization of $N_\Delta((\text{Set}_\Delta^\mathcal{D})^\circ)$. Since \mathbf{MS}^ω and \mathcal{C} are small ∞ -categories, hence the ∞ -category $N_\Delta((\text{Set}_\Delta^\mathcal{D})^\circ)$ is an ∞ -topos. The accessible left exact localization \mathcal{X} is also an ∞ -topos.

Let $\widehat{\mathbf{MCat}}_\infty$ denote the ∞ -bicategory of (not necessary small) motivic ∞ -categories. Let ${}^L\mathbf{MTop}$ be a subcategory of $\widehat{\mathbf{MCat}}_\infty$ whose objects are motivic ∞ -topoi and morphisms are left exact colimit preserve functors. We say that ${}^L\mathbf{MTop}$ is the ∞ -bicategory of motivic ∞ -topoi, and a left exact colimit preserving functor between motivic ∞ -topoi is a *geometric morphism*.

Definition 5.6 (cf. [7] p.369, Definition 5.2.8.8(Joyal)). Let \mathcal{C} be a motivic ∞ -category. A factorization system (S_L, S_R) is a pair of collections of morphisms of \mathcal{C} which satisfy the following axioms:

- (1) The collections S_L and S_R are closed under retracts.
- (2) The collection S_L is left orthogonal to S_R
- (3) For any morphism $h : X \rightarrow Z$ in \mathcal{C} , there exists an object Y of \mathcal{C} , morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ such that $h = g \circ f$, $f \in S_L$ and $g \in S_R$.

Let \mathcal{X} and \mathcal{Y} be motivic ∞ -topoi. Let $\text{Fun}^*(\mathcal{X}, \mathcal{Y})$ denote the full subcategory of $\text{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by those functors $f : \mathcal{X} \rightarrow \mathcal{Y}$ which admit geometric left adjoints.

Definition 5.7 ([10] p.27, Definition 1.4.3). Let \mathcal{K} be a motivic ∞ -topos. A geometric structure on \mathcal{K} is a factorization system $(S_L^\mathcal{X}, S_R^\mathcal{X})$ on $\text{Fun}^*(\mathcal{K}, \mathcal{X})$, which depends functorially on \mathcal{X} . We say that \mathcal{K} is a *classifying motivic ∞ -topos* and a morphism in $S_R^\mathcal{X}$ is a *local morphism*. For any classifying motivic ∞ -topos \mathcal{K} and motivic ∞ -topos \mathcal{X} , we let $\text{Str}_{\mathcal{K}}^{\text{loc}}(\mathcal{X})$ denote the subcategory of $\text{Fun}^*(\mathcal{K}, \mathcal{X})$ spanned by all the objects of $\text{Fun}^*(\mathcal{K}, \mathcal{X})$, and all morphisms of $\text{Str}_{\mathcal{K}}^{\text{loc}}(\mathcal{X})$ are local. We say that an object of $\text{Str}_{\mathcal{K}}^{\text{loc}}(\mathcal{X})$ is a \mathcal{K} -*structured sheaf* on \mathcal{X} . If a geometric morphism $f : \mathcal{K} \rightarrow \mathcal{K}'$ of classifying motivic ∞ -topoi carries all local morphisms on $\text{Fun}^*(\mathcal{K}, \mathcal{X})$ to

local morphisms on $\text{Fun}^*(\mathcal{K}', \mathcal{X})$ for any motivic ∞ -topos, we say that f is *compatible with the geometric structures*.

Let \mathcal{K} be a classifying motivic ∞ -topos. By the motivic scaled straightening and unstraightening Theorem 5.4, we have a Quillen equivalence

$$\text{Mot}(\text{St}^{\text{sc}}) : \text{Mot}(\text{Set}_{\Delta}^+)/{}_{\text{L}\mathbf{M}\mathbf{Top}}^{\text{op}} \rightleftarrows \text{Mot}(\text{Set}_{\Delta}^+)^{\text{sc}}[{}^{\text{L}\mathbf{M}\mathbf{Top}}] : \text{Mot}(\text{Un}^{\text{sc}}).$$

Under the Quillen equivalence, the functor $\text{Str}_{\mathcal{K}}^{\text{loc}} : {}^{\text{L}\mathbf{M}\mathbf{Top}} \rightarrow \widehat{\mathbf{M}\mathbf{Cat}}_{\infty}$ determines a ∞ -category ${}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K})$ and a locally coCartesian fibration $p : {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K}) \rightarrow {}^{\text{L}\mathbf{M}\mathbf{Top}}$. We call the ∞ -category ${}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K})$ the \mathcal{K} -structured motivic ∞ -topoi. Furthermore we have that the motivic ∞ -categorical Yoneda functor $\text{Fun}^*(\mathcal{K}, -) : {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{K}/} \rightarrow \widehat{\mathbf{M}\mathbf{Cat}}_{\infty}$ classifies an ∞ -category ${}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{K}/}$ and a locally coCartesian fibration $q : {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{K}/} \rightarrow {}^{\text{L}\mathbf{M}\mathbf{Top}}$. By the similar argument of the proof of [7, p.610, Proposition 6.3.4.6], we have that the ∞ -category ${}^{\text{R}\mathcal{M}\mathcal{T}\mathbf{op}}$ admits pullbacks. In other words, for any geometric morphism $f : \mathcal{K} \rightarrow \mathcal{K}'$, the forgetful functor $f_* : {}^{\text{R}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{K}} \rightarrow {}^{\text{R}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{K}'}$ admits a right adjoint. Note that for any motivic ∞ -topos \mathcal{X} , the opposite category of ${}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{X}/}$ is weakly equivalent to $({}^{\text{R}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{X}})^{\text{op}}$ as ∞ -categories.

Consider the case that $f : \mathcal{K} \rightarrow \mathcal{K}'$ is a geometric morphism of motivic classifying ∞ -topoi such that f is compatible with the geometric structures. Then we have the (homotopically) commutative diagram of ∞ -categories:

$$\begin{array}{ccc} {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{K}/} & \xrightleftharpoons[f^{-1}]{f_*} & {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}_{\mathcal{K}'}/ \\ \uparrow & & \uparrow \\ {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K}) & \xleftarrow[f^{-1]}{} & {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K}') \end{array}.$$

We will prove that the lower horizontal functor has a left adjoint:

Theorem 5.8. *Let $f : \mathcal{K} \rightarrow \mathcal{K}'$ be a geometric morphism of motivic classifying ∞ -topoi such that f is compatible with geometric structures. Given the commutative diagram*

$$\begin{array}{ccc} {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K}) & \xleftarrow[f^{-1]}{} & {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K}') \\ & \searrow p & \swarrow q \\ & {}^{\text{L}\mathbf{M}\mathbf{Top}} & \end{array}$$

where f^{-1} is the induced functor by f , and p and q are locally coCartesian fibrations. Then $f^{-1} : {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K}') \rightarrow {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K})$ admits a left adjoint relative to ${}^{\text{L}\mathbf{M}\mathbf{Top}}$.

proof. Let $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ be a geometric morphism between motivic ∞ -topoi. Then the functor $f^{-1} : {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K}') \rightarrow {}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K})$ carries locally q -coCartesian edges to locally p -coCartesian edges on ${}^{\text{L}\mathbf{M}\mathbf{Top}}$. In fact, a locally q -coCartesian edge ${}^{\text{L}\mathcal{M}\mathcal{T}\mathbf{op}}(\mathcal{K}')$ forms to $\alpha : \mathcal{O}_{\mathcal{X}} \rightarrow \pi_* \mathcal{O}_{\mathcal{X}}$ for $\mathcal{O}_{\mathcal{X}} \in \text{Str}_{\mathcal{K}'}^{\text{loc}}(\mathcal{X})$. Since α is q -coCartesian, we have a chain of equivalences

$$(\pi_* \mathcal{O}_{\mathcal{X}}) \circ f \simeq (\pi \circ \mathcal{O}_{\mathcal{X}}) \circ f \simeq \pi \circ (\mathcal{O}_{\mathcal{X}} \circ f) \simeq \pi_*(\mathcal{O}_{\mathcal{X}} \circ f).$$

Hence $f^{-1}(\alpha) : \mathcal{O}_X \circ f \rightarrow (\pi_* \mathcal{O}_X) \circ f$ is equivalent to a locally p -coCartesian edge $: \mathcal{O}_X \circ f \rightarrow \pi_*(\mathcal{O}_X \circ f)$.

By [9, Proposition 7.3.2.6], it is sufficient to prove that the functor

$$f_X^{-1} : \mathrm{Str}_{\mathcal{K}'}^{\mathrm{loc}}(\mathcal{X}) \rightarrow \mathrm{Str}_{\mathcal{K}}^{\mathrm{loc}}(\mathcal{X})$$

admits a left adjoint. Let \mathcal{O}_X be a object of $\mathrm{Fun}^*(\mathcal{K}, \mathcal{X})$ and \mathcal{O}'_X an object of $\mathrm{Fun}^*(\mathcal{K}', \mathcal{X})$. Let $\phi : \mathcal{O}_X \rightarrow \mathcal{O}'_X \circ f$ be a local morphism in $\mathrm{Fun}^*(\mathcal{K}, \mathcal{X})$. We can obtain a left Kan extension $f_* \mathcal{O}_X : \mathcal{K}' \rightarrow \mathcal{X}$ along f . The transformation $\phi_* : f_*(\mathcal{O}_X) \rightarrow \mathcal{O}'_X$ induced by ϕ gives a functorial factorization

$$f_*(\mathcal{O}_X) \rightarrow \mathrm{MSpc}_{\mathcal{K}'}^{\mathcal{K}}(\mathcal{O}_X) \xrightarrow{\mathrm{MSpc}(\alpha)} \mathcal{O}'_X$$

where $\mathrm{MSpc}(\alpha)$ is local. Hence we can get a functor $\mathrm{MSpc}_{\mathcal{K}, \mathcal{X}}^{\mathcal{K}'}$ which is a left Kan extension of f_X^{-1} .

Remark 5.9. In this paper, the motivic ∞ -category ${}^{\mathrm{L}}\mathfrak{MTop}(\mathcal{K})$ is constructed by following in [10, Remark 1.4.17].

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